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Disjunctive Permutations and their Applications

We consider disjunctive permutations and the possibility of its application in the graph theory. For example, disjunctive permutations can be applied for constructing expanders that contain this subgraph.

In this paper we consider so-called disjunctive permutations and their application to some problems of the graph theory [1].

Definition 1. Let a group of permutations act on the set U, $A \subset U$, then a permutation $g \in \Gamma$ will be called a disjunctive permutation of the set A, if $g(A) \cap A = \varnothing$.

The following two theorems prove the existence of disjunctive permutations under certain limitations on the number of elements of the subset A of given set U.

Theorem 1 [1]**.** If the group Γ acts transitively on the set U, $|U| = n$, $A \subset U$, $|A| = k$ and the condition $k < \sqrt{n}$ is fulfilled , then there exists a disjunctive permutation of the set A in the group Γ.

Theorem 1 can be easily generalized for the case of not necessarily a transitive group.

Theorem 2. If the group Γ acts on the set U, $|U| = n$, $U = \sum_{i=1}^{s} U_i$ where $\{U_i\}_{i=1}^{s}$ is the set of orbits of this group, $A \subset U$, $|A| = k$ and the following condition is fulfilled

$$\sum_{i=1}^{s} |A \cap U_i|^2 / |U_i| < 1,$$

then there exists a disjunctive permutation of the set A in the group Γ.

An important question is the accuracy of the estimate $k < \sqrt{n}$. Suppose the group Γ acts transitively on the set U, $|U| = n$. What is the maximal possible order k of the subset A, for which there exists a disjunctive permutation?

Below we will show that this estimate can not be improved in the general case.

Namely, one can construct an example of the set U of arbitrarily large order n, in which a certain group Γ acts transitively and in which there exists a subset A of order $\lceil \sqrt{n} \rceil$ for which there exists no disjunctive permutation.

Let's consider the finite Galois projective plane $PG(2,m)$, which contains $n = m^2 + m + 1$ points (and the same number of straight lines) and exists for all $m = p^s$. It is known [3], that a group of automorphisms $PG(2,m)$ contains a cyclic subgroup Z_n, that acts transitively on the set of points and on the set of straight lines of this plane.

Suppose A is any straight line of the given plane, and $g \in Z_n$ is any permutation. Then $A_1 = g(A)$ also is the straight line of this plane and

$$|A| = |A_1| = m+1 = \lceil \sqrt{n} \rceil .$$

But

$$g(A) \cap A \neq \varnothing ,$$

since each pair of the straight lines has exactly one common point.

Let's show how disjunctive permutations can be applied in some problems of the theory of graphs, for example, for constructing expanders that contain this subgraph.

Recall, that the graph $G_n, |G_n^0| = n$ is called an expander, if for any $A \subset G_n^0,\ |A| \leq n/2$, the following inequality is satisfied:

$$\rho(A, G_n) \geq \alpha \, |A| ,$$

where $\alpha > 0$ is a constant [1, 2].

Definition 2 [3]. Let F, J be two graphs, $F^0 \cap J^0 = \varnothing$, $|F^0| = |J^0| = n$ and let $\varphi : F^0 \to J^0$ be some bijection. We denote by $G = F\langle\varphi\rangle J$ the graph, obtained from the graph $F + J$ by identifying each vertex $a \in F^0$ with the vertex $\varphi(a) \in J^0$. The graph G is called the permutation joint of graphs F and J .

Suppose $\tau_1 : F \to G,\ \tau_2 : J \to G$ is the injective embeddings of graphs. Then the mapping $\varphi^* = \tau_1^{-1} \varphi \tau_2$ is a permutation on the set G^0 of vertices of the graph G.

Let's consider an arbitrary connected graph H_n, where $H_n^0 = V_n = \{1, 2, ..., n\}$.

The sets U and A are defined as follows:

$$U = \{X : X \subset V_n,\ |X| \leq n/2\},$$

$$A = \{X : X \in U,\ \rho(X, H_n) < \alpha \, |X|\}.$$

Let now the symmetric group S_n act transitively on the set V_n. Then the group S_n also acts on the set U by the rule:

$$g \in S_n, X \in U \Rightarrow g(X) = \{g(x) : x \in X\}.$$

In this case the sets U_k are the orbits, where

$$U_k = \{X : X \in U, |X| = k\}, \quad k = 1,2,...,s, \; s = \lfloor n/2 \rfloor.$$

Let two graphs F and J be isomorphic to the graph H_n.

Let's consider the graph $G = H\langle\varphi\rangle J$. It is obvious, that if the permutation φ is a disjunctive mapping of the set A, then the graph G will be an expander.

Lemma 1. If H_n is an arbitrary connected graph, then for sufficiently small $\alpha > 0$ the following inequality is satisfied:

$$|A \cap U_k|^2 / |U_k| < (ck/n)^{1-2\alpha},$$

where

$$0 < c < 2.$$

Lemma 2. If

$$0 < c < 2, \; \theta > 0,$$

then

$$\sum_{k=1}^{\lfloor n/2 \rfloor} (ck/n)^{\theta k} = o(1).$$

Theorem 3. The joint $G = H\langle\varphi\rangle J$ of two isomorphic connected graphs by a random permutation will be an expander (for sufficiently small $\alpha > 0$) with probability $1 - o(1)$.

Proof immediately follows from Theorem 2 and Lemmas 1 and 2.

Note that there exists an effective way of checking whether this graph, in particular the resulting graph $G = H\langle\varphi\rangle J$, will indeed be an expander [1,2].

References

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