Fundamental trigonometric interpolation and approximation polynomials and splines

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Abstract. We consider fundamental trigonometric polynomials and splines on uniform grids of two types, which allow to construct interpolation approximations that depend linearly on the values of the interpolated function. The domain of fundamental systems of functions should not be narrower than the interval of definition of the interpolated function. Fundamental trigonometric MLS-polynomials and MLS-splines are also considered on the same grids. They allow you to construct approximations of functions by the method of least squares and also depend linearly on the values of the approximate function. Splines, which in some cases have polynomial analogues, were considered as trigonometric splines. The given material is illustrated by the graphs of fundamental trigonometric polynomials and splines on the uniform grid with different parameter values. The considered fundamental trigonometric polynomials and splines can be recommended for application in many problems related to the approximation of functions, in particular, in the problems of mathematical modeling of signals and processes.

1. Introduction

Approximation that is representation of a known or unknown function through a set of some special functions can be considered as a central theme of analysis. Such special functions are well defined, easy to calculate and have certain analytical properties [1]. Algebraic and trigonometric polynomials, exponential functions [2], polynomial [3-7] and trigonometric [8] splines, etc. often act as special functions.

Among such functions, systems of functions fundamental on uniform grids attract special attention.

Consideration of systems of fundamental functions on the grid of functions we will start with the interpolation problem.

Let some grid Δ_N, $\Delta_N = \{t_i\}_{i=0}^{N}$, $0 \le t_0 < t_1 < \dots < t_N \le 1$ be given on the segment $[0, T]$. Suppose that the function $f(t)$ is also given on the segment $[0, T]$, and the exact values $f(t_j) = f_j$, $(j = 0, 1, \dots, N)$ of this function in the grid nodes are known. We need to write a generalized polynomial

$$\Phi_N(t) = c_0\varphi_0(t) + c_1\varphi_1(t) + \dots + c_N\varphi_N(t)$$

According to the system of linearly independent functions $\varphi_0(t), \varphi_1(t), ... \varphi_N(t)$. This polynomial depends on the $N+1$ parameters $c_0, c_1, \dots, c_N$ and satisfies the conditions

$$\Phi_N\left(x_j\right) = f_i, \quad j = 0,1,\dots,N.$$

The formulated problem is called the interpolation problem [9,10], and the polynomial $\Phi_N(x)$ is called the generalized interpolation polynomial.

In many cases in interpolation problems a system of functions $\varphi_0(t), \varphi_1(t), ..., \varphi_N(t)$ on the grid Δ_N should be chosen so that the coefficients c_k, ($k = 0,1,...,N$) are the values of the interpolated function $f(t)$ in the nodes of this grid. In this case, the polynomial $\Phi_N(t)$ is

$$\Phi_N(t) = f_0\varphi_0(t) + f_1\varphi_1(t) + ... + f_N\varphi_N(t). \tag{1}$$

It is clear that in this case the relations

$$\varphi_j(t_i) = \begin{cases} 1, & j = i; \\ 0, & j \neq i. \end{cases} \tag{2}$$

must be satisfied for the functions $\varphi(t)$.

Functions for which relations (2) take place on the grid Δ_N are called fundamental functions on this grid [11]. The domain of definition of fundamental systems of functions should not be narrower than the interval of setting the interpolated function.

Representation of interpolation polynomials in the form of (1) has significant advantages over other forms of representation. For example, there is no need to calculate the coefficients c_k of the interpolation polynomial. The values of the functions $\varphi_0(t), \varphi_1(t), ..., \varphi_N(t)$ can be calculated in advance. Further, representation (1) linearly depends on the values of the function f_j, $(j = 0,1,\dots,N)$, which in many cases plays a significant role. Finally, the construction of interpolation polynomials of several variables is greatly simplified, because the fundamental functions of several variables are the product of the fundamental functions for each variable. The most well-known fundamental systems of functions today include the system of fundamental Lagrange interpolation functions, the system of fundamental interpolation trigonometric polynomials, and the system of Kotelnikov and Shannon functions. However, there are other systems of fundamental functions [12].

We consider fundamental trigonometric polynomials and splines given on uniform grids. Note that trigonometric splines were considered in [13].

2. System of fundamental trigonometric interpolation polynomials

Suppose that $T \equiv 2\pi$. In this segment we will consider uniform grids $\Delta_N^{(I)} = \{t_j^{(I)}\}_{j=1}^N$, ($I = 0,1$), where $t_j^{(0)} = \dfrac{2\pi}{N}(j-1)$, $t_j^{(1)} = \dfrac{\pi}{N}(2j-1)$, $N = 2n-1$, ($n = 1, 2, ...$).

It is known [14], that fundamental on these grids trigonometric polynomials $tm_k\left(t\right)$ ($k = 1, 2, \cdots, N$) of order n can be written as follows

$$tm_k^{(I)}\left(t\right) = \frac{1}{N}\left[1 + 2\sum_{j=1}^{n}\cos j\left(t - t_k^{(I)}\right)\right].$$

Graphs of some fundamental trigonometric polynomials on the grid $\Delta_N^{(0)}$ are given in Figure 1. Note that here and in the following $N = 9$.

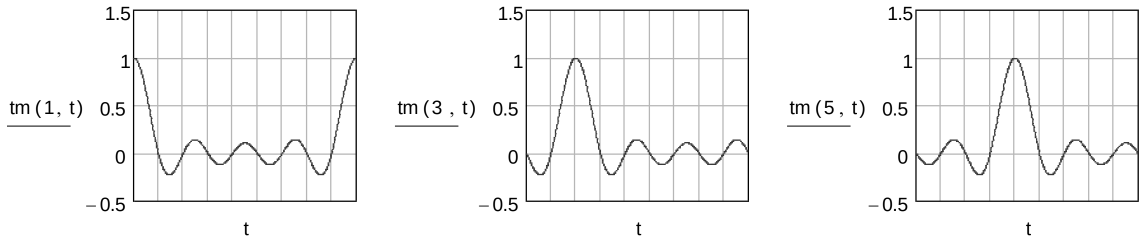

Figure 1. Fundamental on the grid $\Delta_N^{(0)}$ trigonometric polynomials $tm_1(t), tm_3(t)$ and $tm_5(t)$.

Graphs of some fundamental trigonometric polynomials on the uniform grid $\Delta_N^{(1)}$ are given in Figure 2. Note, that the nodes of grid $\Delta_N^{(1)}$ are located between the nodes of grid $\Delta_N^{(0)}$ and are not displayed on the graph.

Using a system of fundamental trigonometric polynomials $tm_k^{(I)}(t)$, $k = 1, 2, \cdots, N$, an interpolation trigonometric polynomial can be written as

$$T_n^{(I)}(t) = \sum_{k=1}^{N} f_k^{(I)} tm_k^{(I)}(t).$$

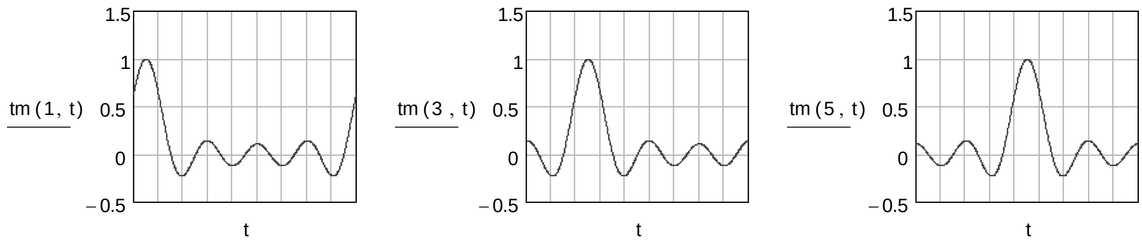

Figure 2. Fundamental on the grid $\Delta_N^{(1)}$ trigonometric polynomials $tm_1(t), tm_3(t)$ and $tm_5(t)$.

3. System of fundamental trigonometric interpolation splines

Trigonometric interpolation splines on the grids Δ_N^I were considered in [12, 13]. In this paper we consider only those trigonometric splines that have polynomial analogues.

Fundamental trigonometric interpolation splines can be written as [15]

$$ts_j^{(I)}(\sigma, r, N, t) = \frac{1}{N}\left\{1 + 2\sum_{k=1}^{\frac{N-1}{2}} \frac{C_k^{(I)}(\sigma, r, N, j, t)}{H_k^{(I)}(r, N)}\right\};$$

where

$$C_k^{(I)}(\sigma, r, N, j, t) = \sigma_k(r)\cos k\left(t - t_j^{(I)}\right) +$$

$$+\sum_{m=1}^{\infty}(-1)^{mI}\left[\sigma_{mN+k}(r)\cos(mN+k)\left(t - t_j^{(I)}\right) + \sigma_{mN-k}(r)\cos(mN-k)\left(t - t_j^{(I)}\right)\right];$$

$$H_k^{(I)}(r, N) = \sigma_k(r) + \sum_{m=1}^{\infty}(-1)^{mI}\left[\sigma_{mN+k}(r) + \sigma_{mN-k}(r)\right];$$

$$\sigma_k(r,N)=\left[\frac{\sin(.5hk)}{k}\right]^{1+r}, h=\frac{2\pi}{N}.$$

Graphs of fundamental trigonometric splines $ts_k^{(0)}(r,t)$ at different values of the parameter r are given in Figure 3.

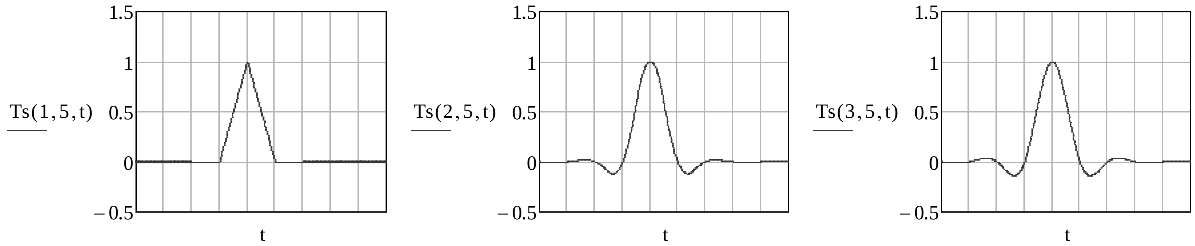

Figure 3. Fundamental on the grid $\Delta_N^{(0)}$ trigonometric splines $ts_j^{(0)}(r,t)$ for $j=5$ and parameter values $r=1,2,3$.

Note that such trigonometric splines with odd values of this parameter have polynomial analogues which are simple polynomial splines of odd degree. At even values of the parameter the polynomial analogues of these splines are unknown.

Note that here and in the future in the notation on the graphs, we will omit the fixed parameters. For example, since σ and the number of grid nodes $N=9$ has already been determined, we do not indicate the dependence on these parameters on the graphs.

Fundamental trigonometric splines $ts_k^{(1)}(\sigma,r,t)$ have no polynomial analogues at odd values of the parameter r. And at even values of the parameter polynomial analogues are simple polynomial splines of even degree. Graphs of such splines at different values of the parameter r are shown in Figure 4.

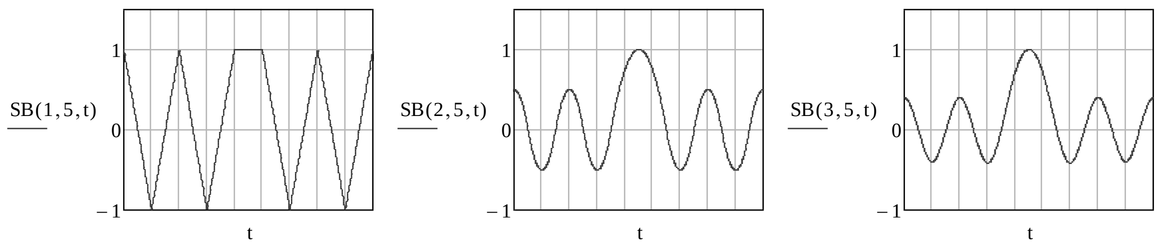

Figure 4. Fundamental on the grid $\Delta_N^{(1)}$ trigonometric splines $ts_j^{(1)}(r,t)$ for $j=5$ and parameter values $r=1,2,3$.

Using fundamental trigonometric interpolation splines $ts_j^{(I)}(r,t)$, interpolation trigonometric spline $St^{(I)}(r,t)$, that interpolates function $f(t)$ in nodes of grids $\Delta_N^{(I)}$, can be represented as

$$St^{(I)}(r,t)=\sum_{k=1}^{N} f_k^{(I)} ts_k^{(I)}(r,t).$$

Fundamental trigonometric splines are orthogonal only in the sense of a discrete scalar product, ie

$$\sum_{k=1}^{N} ts_i^{(I)}(t_k^{(I)})\, ts_j^{(I)}(t_k^{(I)})=\begin{cases}1, & i=j;\\ 0, & i\neq j.\end{cases} \quad i,j=1,2,\dots,N.$$

4. Approximation of functions by the method of least squares

In many cases, instead of interpolating the functions given by their values in the nodes of a grid, it is advisable to use their approximation by the method of least squares (MLS) [2]. As we know, such problem is formulated as follows.

On the segment $[0,T]$ consider the grids $\Delta_N^{(I)}$ and some function $f(t)$. Moreover, the values of this function $f(t_j)=f_j$, $(j=0,1,\dots,N)$ in the nodes of the grid are known. It is necessary to construct a generalized polynomial

$$\Phi_N(t)=c_0\varphi_0(t)+c_1\varphi_1(t)+\ldots+c_N\varphi_N(t)$$

according to the system of linearly independent functions $\varphi_0(t),\varphi_1(t),\ldots\varphi_N(t)$, which depends on q ($q\le N$), parameters $c_0,c_1,\dots,c_q$, on which the minimum value is reached

$$E_N^{(I)}(f,T_q)=\sum_{j=0}^{N}\left[f(t_j^{(I)})-T_q(t_j^{(I)})\right]^2. \tag{3}$$

It is known that together with the classical Fourier series theory (see, for example, [15]), which considers the continuous case of criterion (3), there is a theory that considers functions on a discrete set of equidistant points. it is clear, that there are finite Fourier series [2]. In this case, the best MLS approximation is provided by the segment of finite Fourier series. We will use this fact when constructing the required trigonometric polynomials.

Note that the interpolation trigonometric polynomial $T_n^{(I)}(t)$, that interpolates function $f(t)$ in grid nodes $\Delta_N^{(I)}$, can be written as

$$T_n^{(I)}(t)=\frac{a_0^{(I)}}{2}+\sum_{k=1}^{n}\left(a_k^{(I)}\cos kt+b_k^{(I)}\sin kt\right). \tag{4}$$

Coefficients $a_0^{(I)},a_k^{(I)},b_k^{(I)}$, $(k=1,2,\ldots,n)$ are calculated by formulas

$$a_0^{(I)}=\tfrac{2}{N}\sum_{j=1}^{N}f_j^{(I)};$$

$$a_k^{(I)}=\tfrac{2}{N}\sum_{j=1}^{N}f_j^{(I)}\cos kt_j^{(I)};\qquad a_k^{(I)}=\tfrac{2}{N}\sum_{j=1}^{N}f_j^{(I)}\sin kt_j^{(I)}. \tag{5}$$

Since the system of functions 1, $\cos kt_j^{(I)}$, $\sin kt_j^{(I)}$ is an orthogonal system in the sense of a discrete scalar product [2], expression (4) can be considered as a finite Fourier series with coefficients (5). It follows from the general theory of Fourier series that the partial sums of finite series (4) provide the best approximation for MLS in the sense that for any q ($q\le n$), the minimum value of $E_N(f,T_q)$ is reached on the polynomial

$$T_q^{(I)}(t)=\frac{a_0^{(I)}}{2}+\sum_{k=1}^{q}\left(a_k^{(I)}\cos kt+b_k^{(I)}\sin kt\right) \tag{6}$$

with coefficients (5). It is clear that when $q=n$ magnitude $E_N(f,T_N)=0$.

Henceforth, for reduction, fundamental trigonometric polynomials and fundamental trigonometric splines approximating a function on a discrete set of points by MLS will be called fundamental trigonometric MLS-polynomials and fundamental trigonometric MLS-splines, respectively.

As before, the task of constructing systems of fundamental trigonometric MLS-polynomials and fundamental trigonometric MLS-splines is urgent. Let's consider this problem in more detail.

5. System of fundamental trigonometric MLS-polynomials

We will construct such a system as follows. Substituting expressions (5) for the coefficients of the polynomial $T_q^{(I)}(t)$ in (6), after changing the order of summation we obtain

$$T_q^{(I)}(t) = \frac{1}{2}\frac{2}{N}\sum_{j=1}^{N} f_j^{(I)} +$$

$$+\sum_{k=1}^{q}\left[\frac{2}{N}\sum_{j=1}^{N} f_j^{(I)} \cos kt_j^{(I)} \cos kt + \frac{2}{N}\sum_{j=1}^{N} f_j^{(I)} \sin kt_j^{(I)} \sin kt\right] =$$

$$= \sum_{j=1}^{N} f_j^{(I)} \left\{\frac{1}{N} + \frac{2}{N}\sum_{k=1}^{q}\left[\cos kt_j^{(I)} \cos kt + \sin kt_j^{(I)} \sin kt\right]\right\} =$$

$$= \sum_{j=1}^{M} f_j^{(I)} \left\{\frac{1}{N}\left[1 + 2\sum_{k=1}^{q}\cos k\left(t - t_j^{(I)}\right)\right]\right\}.$$

This expression can be represented as

$$T_n^{(I)}(t) = \sum_{j=1}^{N} f_j^{(I)} \varphi_{j,q}^{(I)}(t), \tag{7}$$

where

$$\varphi_{j,q}^{(I)}(t) = \frac{1}{N}\left[1 + 2\sum_{k=1}^{q}\cos k\left(t - t_j^{(I)}\right)\right], \tag{8}$$

$$j = 1, \ldots, N\,; q \le n\,.$$

Therefore the system of fundamental trigonometric MLS-polynomials consists of polynomials $\varphi_{j,q}^{(I)}(t)$, defined by expression (8). Graphs of some MLS-polynomials $\varphi_{j,q}^{(I)}(t)$ on the grids $\Delta_N^{(I)}$ at fixed j and N for different parameter values q are shown in Figures 5, 6.

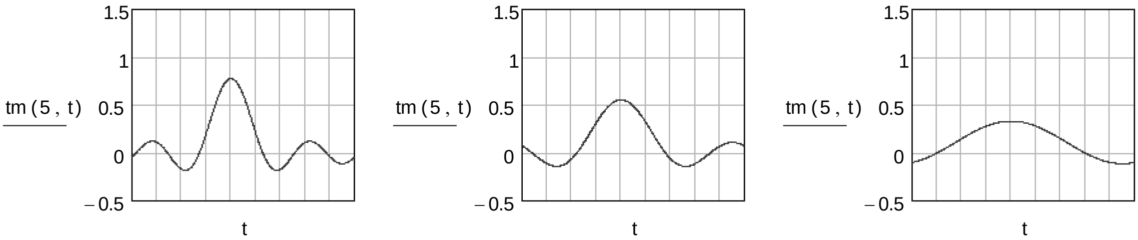

Figure 5. Fundamental trigonometric MLS-polynomials on the grid $\Delta_N^{(0)}$; $j = 5, q = 3,2,1$.

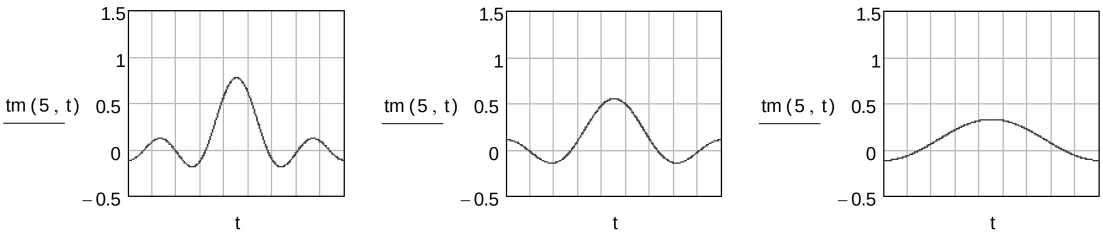

Figure 6. Fundamental trigonometric MLS-polynomials on the grid $\Delta_N^{(1)}$; $j = 5, q = 3,2,1$.

6. **System of fundamental trigonometric MLS-splines**

A certain disadvantage of the polynomial of the order m of the best RMS approximation, construct on the values of the function on a discrete set of equidistant points, is its analyticity. This polynomial is not very convenient for approximating functions with low smoothness as a result. To approximate such functions, it is more convenient to use trigonometric splines, the values of which in the grid nodes coincide with the values of the trigonometric MLS-polynomial of order q constructed from the values of the function on a discrete set of equidistant points.

It is easy to show that a trigonometric spline of order q whose values in the grid nodes coincide with the values of the polynomial of discrete approximation of the same order, constructed from the values of the function on a discrete set of equidistant points, can be represented as

$$Ts_{r,q}^{(I)}(x) = \sum_{j=1}^{N} f_j ts_{j,q}^{(I)}(r, x),$$

where

$$ts_{j,q}^{(I)}(r,t) = \frac{1}{N}\left\{1 + 2\sum_{k=1}^{q} \frac{C_k^{(I)}(r,N,j,t)}{H_k^{(I)}(r,N)}\right\}.$$

Functions $C_k^{(I)}(\sigma,r,N,j,t)$, constants $H_k^{(I)}(r,N)$ and $\sigma_k(r,N)$ are determined as before.

Trigonometric splines $ts_{j,q}^{(I)}(r, x)$ will be called fundamental trigonometric MLS-splines. Graphs of some fundamental trigonometric MLS-splines $ts_{j,q}^{(I)}(r,x)$ on the grids $\Delta_N^{(I)}$ with fixed j for different values of parameters q and r are shown in Figures 7-10.

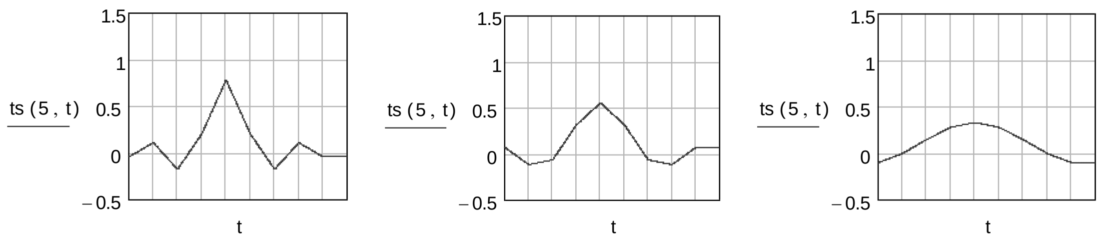

Figure 7. Fundamental trigonometric MLS-splines on the grid $\Delta_N^{(0)}$; $r=1$; $q=3,2,1$.

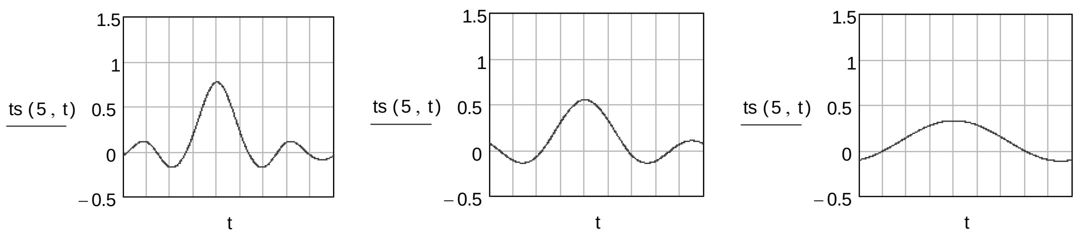

Figure 8. Fundamental trigonometric MLS-splines on the grid $\Delta_N^{(0)}$; $r=3$; $q=3,2,1$.

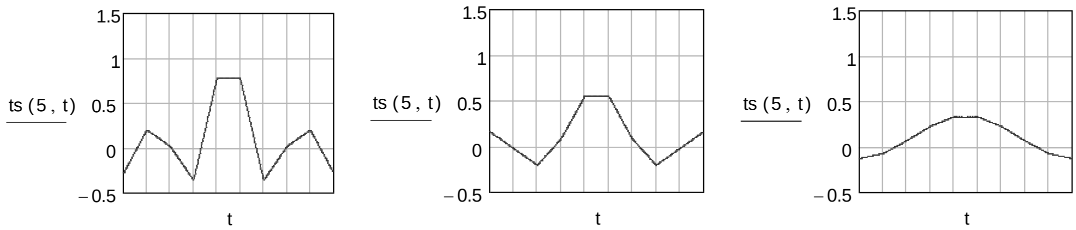

Figure 9. Fundamental trigonometric MLS-splines on the grid $\Delta_N^{(1)}$; $r=1$; $q=3,2,1$.

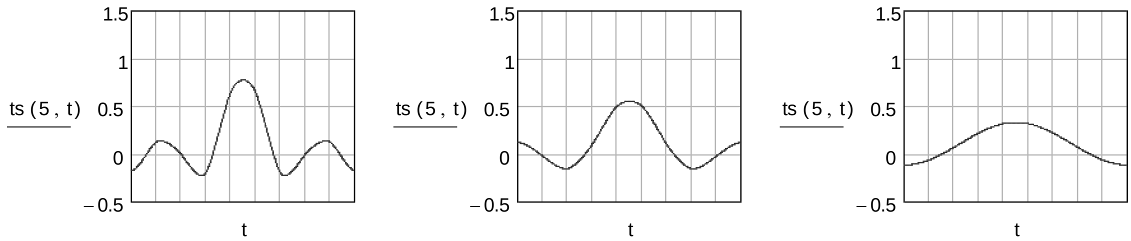

Figure 10. Fundamental trigonometric MLS-splines on the grid $\Delta_N^{(1)}$; $r=2$; $q=3,2,1$.

Note that as can be seen from Figures 6,8, the trigonometric MLS-spline on the grids $\Delta_N^{(I)}$ at the value of the parameter $r=1$ consists of broken, which are sewn together in the nodes of the grid $\Delta_N^{(0)}$.

Note that the fundamental trigonometric MLS-polynomials and MLS-splines lose the properties of orthogonality in the continuous and discrete sense.

Conclusions

Fundamental trigonometric interpolation polynomials and splines on uniform grids $\Delta_N^{(I)}$, ($I=0,1$) are constructed.

Fundamental trigonometric MLS-polynomials and MLS-splines on uniform grids $\Delta_N^{(I)}$, ($I=0,1$) are constructed. They are used to approximate functions by the method of least squares.

It is of interest to construct fundamental trigonometric interpolation polynomials and splines on uniform grids $\Delta_N^{(I)}$, ($I=0,1$) for other cases considered in [15].

Given fundamental trigonometric polynomials and splines can be recommended for application in many problems related to the approximation of functions. In particular, they are used in the problems of mathematical modeling of signals and processes.

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