New reference model of rotation for orientation algorithms analysis

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Abstract. A new analytical representation of the orientation quaternion of a rigid body rotation based on four sequentional turns by simultaneously changing in time angles is proposed. A reference model of rotation, which includes analytical expressions for the orientation quaternion and the quasi-coordinates, is developed for the accuracy estimation of algorithms of orientation in strapdown inertial navigation systems. Numerical simulation of the reference model is performed and trajectories in the configuration space of orientation parameters are build. It is shown that, in comparison with the case of regular precession, the proposed model can be considered as a more generalized case of rotational motion. This model was used to analyze the accumulated errors of the Miller orientation algorithm. Numerical analysis for the drift error of the Miller orientation algorithm is done for different values of coefficients. It is shown, that the Miller algorithm with a new set of coefficients allows a smaller cumulative drift error in comparison with the standard algorithm and the optimized for conic motion algorithm.

1. Inroduction

The problem of the accuracy estimation for orientation finding in strapdown inertial navigation systems (SINS), when the orientation quaternions are calculated on each time of calculation by using a special algorithm, is considered. The ideal signals from angular velocity sensors in form of the quasi-coordinates are used as input for this algorithm [1]:

$$\theta^*_{ni} = \int_{t_{n-1}}^{t_n} \omega_i(t)dt, \quad i = 1,2,3, \tag{1}$$

where $\omega_i(t)$, $i = 1,2,3$ are projections of the vector of the object absolute angular velocity $\vec{\omega}$ on the coordinate system axes. A large number of based on different approaches algorithms of orientation quaternions finding by using initial information are developed [2–4] as of today. One of those algorithm types is based on the use of intermediate parameters such as the orientation vector and corresponding trigonometric expansions in terms of an Euler half-angle. Significant contribution into the development of the orientation vector finding algorithms of different orders is made by A Panov [5, 6]. He developed a number of algorithms that use initial information (1), which is associated with different times of output information. The disadvantages of these algorithms are the need of preliminary calculations to initialize the first step and the large load of an autonomous computing device on every time of computation. Nowadays, in the context of the modern developing of a navigation system instrumentation, increasing of a processing speed of computational devices and application of these devices to high dynamic objects, high interest is focused on the algorithms that use inertial information within a time of computation. These algorithms are based on a polynomial approximation of the angular velocity within a time of computation [7,8]. E.g. the famous Miller algorithm [7] utilizes the quadratic model of the angular velocity which yield the following expression for the orientation vector increment on a time of computation $[t_{n-1}, t_n]$:

$$\vec{\theta}_n = \vec{\theta}_n^* + \alpha(\vec{\theta}_n^1 \times \vec{\theta}_n^3) + \beta\vec{\theta}_n^2 \times (\vec{\theta}_n^3 - \vec{\theta}_n^1), \tag{2}$$

where $\vec{\theta}_n^* = (\theta_{n1}^*, \theta_{n2}^*, \theta_{n3}^*)$, $\vec{\theta}_n^1 = \int\limits_{t_{n-1}}^{t_{n-1}+1/3\Delta T} \vec{\omega}(t)dt$, $\vec{\theta}_n^2 = \int\limits_{t_{n-1}+1/3\Delta T}^{t_{n-1}+2/3\Delta T} \vec{\omega}(t)dt$, $\vec{\theta}_n^3 = \int\limits_{t_{n-1}+2/3\Delta T}^{t_{n-1}+\Delta T} \vec{\omega}(t)dt$ are the output signals of giroscopes obtained within a time of computation on the steps of initial information reading $t_{n-1} + 1/3\Delta T$, $t_{n-1} + 2/3\Delta T$, $t_{n-1} + \Delta T$, ΔT is a time of computation size. The following values are used in the Miller algorithm: $\alpha = \frac{33}{80}$, $\beta = \frac{57}{80}$.

The estimation of the algorithms regular error is based on the special rigid body test motions, which possess the well-known analytically derived orientation vector and the angular velocity vector. An analytical model of the conic motion (the so-called “SPIN-CONE” model) is presented in [9]. Due to the fact, that the local or the cumulative algorithm error has an analytical representation in case of the test motions, algorithms of orientation can be optimized for every specific test motion. Miller proposed in 1983 the optimization of algorithms of orientation in case of the conic motion by only coefficient fitting without changes in the algorithm structure. This optimization procedure is based on the analytical representation of the algorithm error in terms of polynomial series with further obtaining of the unknown coefficients α β, when the series are truncated. Another approach, which is basically identical, was proposed by Ignagni in 1990 [10]. The following coefficient values are used in the optimized for the conic motion Miller algorithm: $\alpha = \frac{36}{80}$, $\beta = \frac{54}{80}$. The optimization methodology for the case of the regular precession and the conic motion was introduced by A Panov [11-13]. This methodology is based on a minimization of the asymptotic representation of the numerical drift error. The improved optimization methodology is presented in [14, 15]. These works contain also results of the algorithm optimization study in case of the generalized conic motion [10], in case of the regular precession and a random angular motion.

The accuracy estimation of algorithms of orientation (also of those, which are optimized for a one specific motion) on more complex rotational motions have a high practical importance, because the conic motion and the regular precession are the quite specific cases of an angular motion of a rigid body, possessing, however, analytical solutions of dynamic and cinematic equations. The new test motions of a rigid body on the base of a trigonometric representation of the orientation quaternions are presented in [16-18].

In the current work, a new reference model of a rigid body rotation based on a four-frequency representation of the orientation quaternions is presented as one of the types of these generalized motions.

2. The reference model, which is based on a sequence of four rotations of a rigid body

Let the first three rotations of a rigid body be performed in order of the Krylov angles φ, ψ and ϑ [2]. The fourth rotation is performed around the second rotated axis by the angle χ. The components of the resulting quaternion are represented then as:

$$\lambda_0 = \tilde{n}os\frac{\chi}{2} \cdot (\cos\frac{\varphi}{2} \cdot \cos\frac{\psi}{2} \cdot \cos\frac{\vartheta}{2} + \sin\frac{\varphi}{2} \cdot \sin\frac{\psi}{2} \cdot \sin\frac{\vartheta}{2}) - \sin\frac{\chi}{2} \cdot (\cos\frac{\varphi}{2} \cdot \sin\frac{\psi}{2} \cdot \cos\frac{\vartheta}{2} +$$

$$+\sin\frac{\varphi}{2}\cdot\cos\frac{\psi}{2}\cdot\sin\frac{\vartheta}{2});$$

$$\lambda_1 = ños\frac{\chi}{2}\cdot(\cos\frac{\varphi}{2}\cdot\cos\frac{\psi}{2}\cdot\sin\frac{\vartheta}{2}-\sin\frac{\varphi}{2}\cdot\sin\frac{\psi}{2}\cdot\cos\frac{\vartheta}{2})-\sin\frac{\chi}{2}\cdot(\sin\frac{\varphi}{2}\cdot\cos\frac{\psi}{2}\cdot\cos\frac{\vartheta}{2}-$$

$$-\cos\frac{\varphi}{2}\cdot\sin\frac{\psi}{2}\cdot\sin\frac{\vartheta}{2});$$

$$\lambda_2 = ños\frac{\chi}{2}\cdot(\cos\frac{\varphi}{2}\cdot\sin\frac{\psi}{2}\cdot\cos\frac{\vartheta}{2}+\sin\frac{\varphi}{2}\cdot\cos\frac{\psi}{2}\cdot\sin\frac{\vartheta}{2})+\sin\frac{\chi}{2}\cdot(\cos\frac{\varphi}{2}\cdot\sin\frac{\psi}{2}\cdot\cos\frac{\vartheta}{2}+$$

$$+\sin\frac{\varphi}{2}\cdot\sin\frac{\psi}{2}\cdot\sin\frac{\vartheta}{2});$$

$$\lambda_3 = ños\frac{\chi}{2}\cdot(\sin\frac{\varphi}{2}\cdot\cos\frac{\psi}{2}\cdot\cos\frac{\vartheta}{2}-\cos\frac{\varphi}{2}\cdot\sin\frac{\psi}{2}\cdot\sin\frac{\vartheta}{2})+\sin\frac{\chi}{2}\cdot(\cos\frac{\varphi}{2}\cdot\cos\frac{\psi}{2}\cdot\sin\frac{\vartheta}{2}-$$

$$-\sin\frac{\varphi}{2}\cdot\sin\frac{\psi}{2}\cdot\cos\frac{\vartheta}{2}). \quad (3)$$

The components of the angular velocity vector are obtained from the inverted kinematic equation for the normalized quaternion $\Lambda(t)$:

$$\vec{\omega}(t) = 2\tilde{\Lambda}(t)\circ\dot{\Lambda}(t), \quad (4)$$

where $\tilde{\Lambda}(t)$ is the conjugated quaternion. Next, we obtain:

$$\omega_1 = -\sin\chi\cdot(\dot{\varphi}\cdot\cos\psi\cdot\cos\vartheta-\dot{\psi}\cdot\sin\vartheta)+\cos\chi\cdot(\dot{\vartheta}-\dot{\varphi}\cdot\sin\psi);$$

$$\omega_2 = \dot{\chi}+\frac{1}{2}\dot{\varphi}\cdot(\sin(\vartheta+\psi)+\sin(\vartheta-\psi))+\dot{\psi}\cdot\cos\vartheta;$$

$$\omega_1 = \sin\chi\cdot(\dot{\vartheta}-\dot{\varphi}\cdot\sin\psi)+\cos\chi\cdot(\dot{\varphi}\cdot\cos\psi\cdot\cos\vartheta-\dot{\psi}\cdot\sin\vartheta). \quad (5)$$

The components of the orientation quaternion (3) and the components of the angular velocity vector (5) can be interpreted as the solution of the dynamic and the kinematic equations of a rigid body rotation jointly. The basis of the reference model are the analytical relations for the orientation quaternion (3) and the quasi-coordinates (1) under considerations that angles φ, ψ, ϑ, χ are known. These angles can be defined on the basis of the existing limitations of the angular velocity of an object considered as a rigid body, or these angles can be obtained as an approximation of a real motion of a particular object.

Assuming that the angles are linear functions of time $\varphi = k_1 t$, $\psi = k_2 t$, $\vartheta = k_3 t$, $\chi = k_4 t$, the relation (5) leads to the analytical representations for the quasi-coordinates (1). To this end we need firstly to find the components of the seeming turn vector

$$\vec{\theta}(t) = (\theta_1(t), \theta_2(t), \theta_3(t)) = \int_0^t \vec{\omega}(t)dt, \quad i = 1,2,3 \quad (6)$$

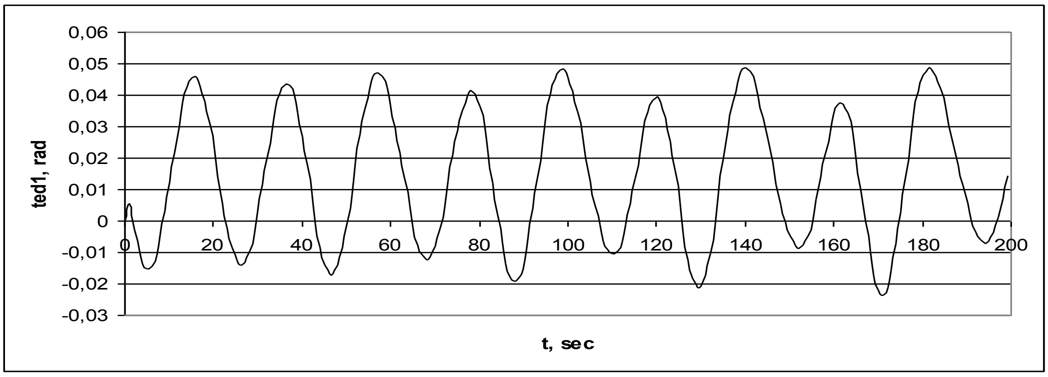
and then to apply the formula:

$$\theta^*_{ni} = \theta_i(t_n) - \theta_i(t_{n-1}), \ i = 1,2,3. \quad (7)$$

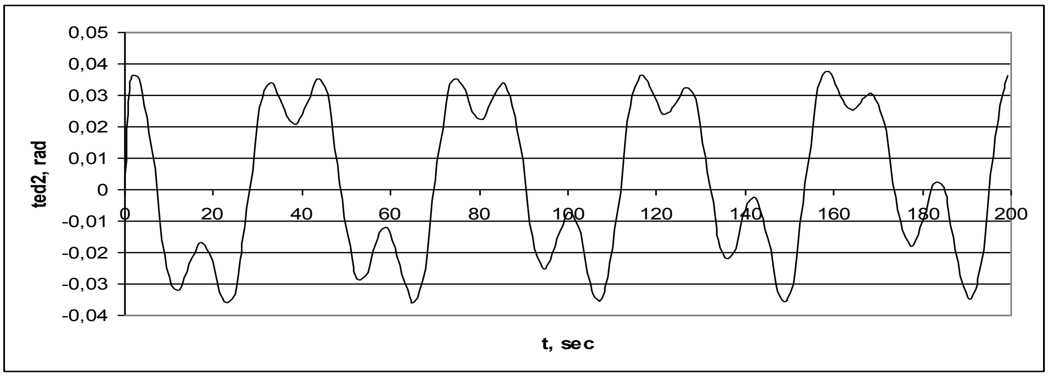
Note that the proposed reference model can be used to model a rotation of a rigid body in case of vibrations, if the corresponding frequencies are properly defined and considering, e.g. $\chi = -\psi$.

3. Numerical Simulation of the four frequency reference model of rotation

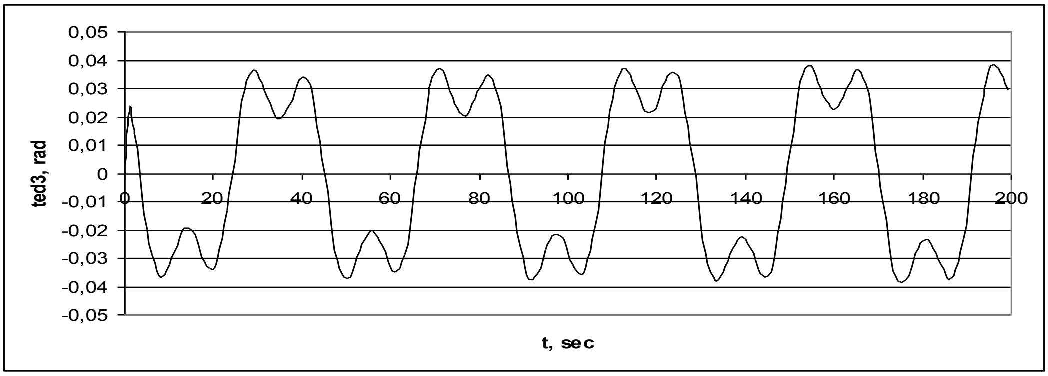
Let us simulate the proposed four-frequency model for some frequency values k_i, $i = 1,2,3,4$. Let us consider the case $\chi = -\psi$. Figure 1 demonstrate the dependence of the quasi-coordinate θ_{ni}^{*} on time for the following frequencies: $k_1 = 0.15$, $k_2 = 1.55$, $k_3 = 0.35$, $k_4 = -1.55$. The obtained trajectories in the configuration space of the orientation parameters are presented in figure 2.



a



b



c

Figure 1. The dependence of the quasi-coordinates on time for the four-frequency reference model considering $k_1 = 0.15$, $k_2 = 1.55$, $k_3 = 0.35$, $k_4 = -1.55$. a – the first axis, b – the second axis, c – the third axis.

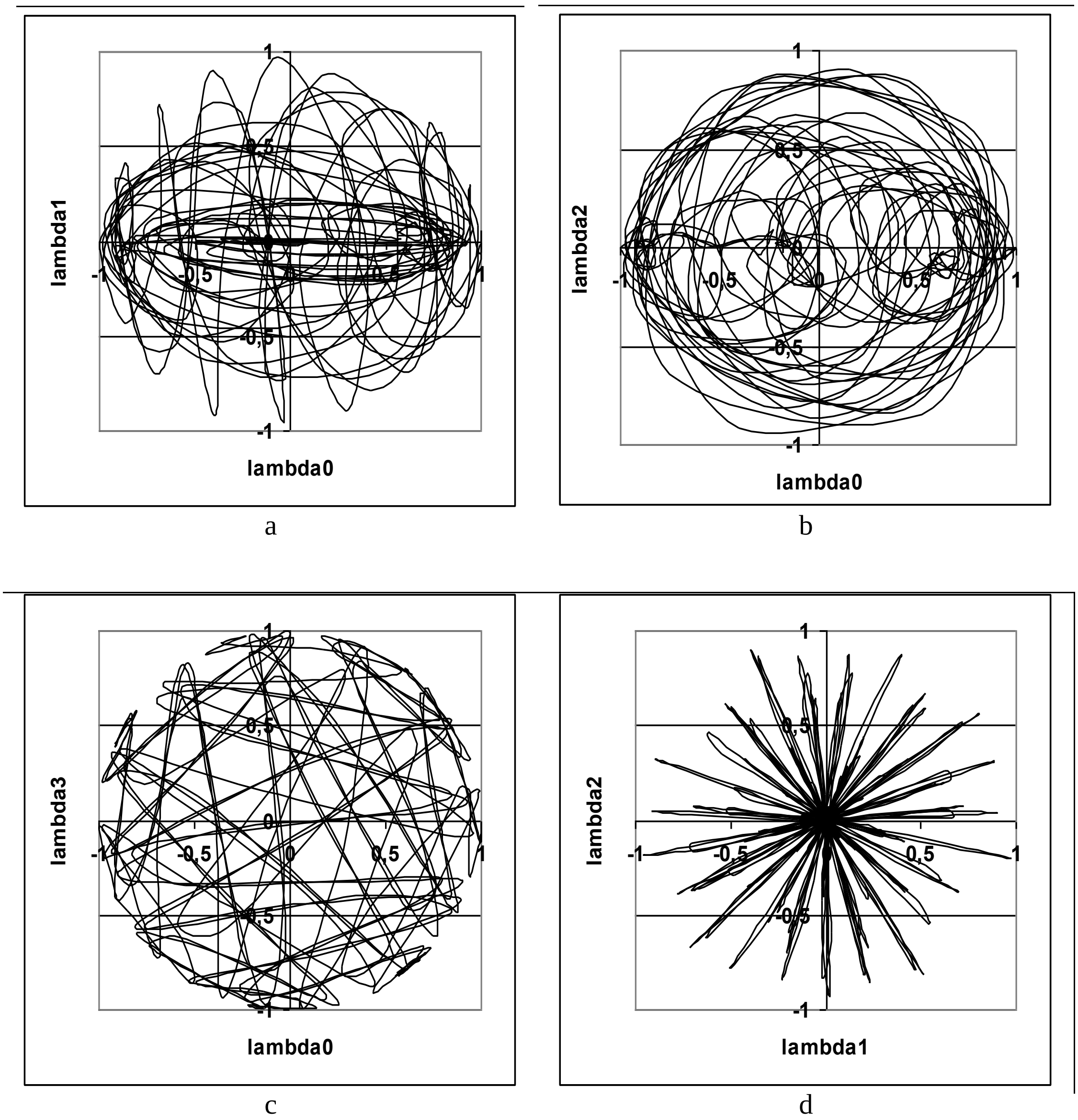


Figure 2. Trajectories in the configuration space for the four-frequency reference model considering $k_1 = 0.15$, $k_2 = 1.55$, $k_3 = 0.35$, $k_4 = -1.55$. a – $\lambda_1(\lambda_0)$; b – $\lambda_2(\lambda_0)$; c – $\lambda_3(\lambda_0)$; d – $\lambda_2(\lambda_1)$.

4. Accuracy study of the Miller algorithm using the four-frequency reference model

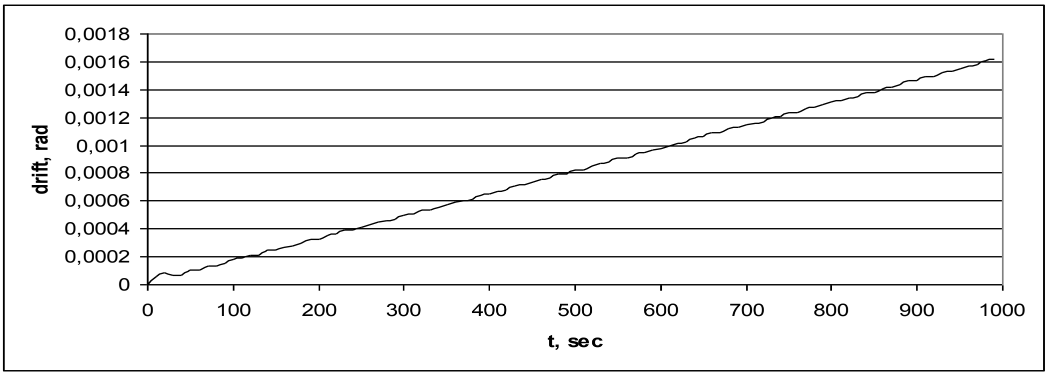
Let us apply the four-frequency reference model to estimate accuracy of the fourth order algorithm of finding the orientation quaternion, where the orientation vector increment on step $[t_{n-1}, t_n]$ is obtained by using the Miller formula (2) for different values of the coefficients α, β. In order to estimate the algorithm accuracy we consider as the error the non-removable orientation residual – the cumulative drift of the computed triad of the axes with respect to its true position defined from the reference model. To this end let us use the drift determination methodology proposed in [2].

Figure 3 *a* shows the dependence of the drift error on time in the interval $t \in [0, 1000]$ sec., which is obtained using the time step $\Delta t = 0.1$ sec for the four-frequency reference model with $k_1 = 0.15$, $k_2 = 1.55$, $k_3 = 0.35$, $k_4 = -1.55$ and the Miller algorithm (2) considering $\alpha = \dfrac{33}{80}$, $\beta = \dfrac{57}{80}$. For the purpose of comparison the fig. 3 b demonstrates the dependence of the drift error on time in case of the optimized Miller algorithm with $\alpha = \frac{36}{80}$, $\beta = \frac{54}{80}$. Note that the optimized algorithm results in a smaller drift error.

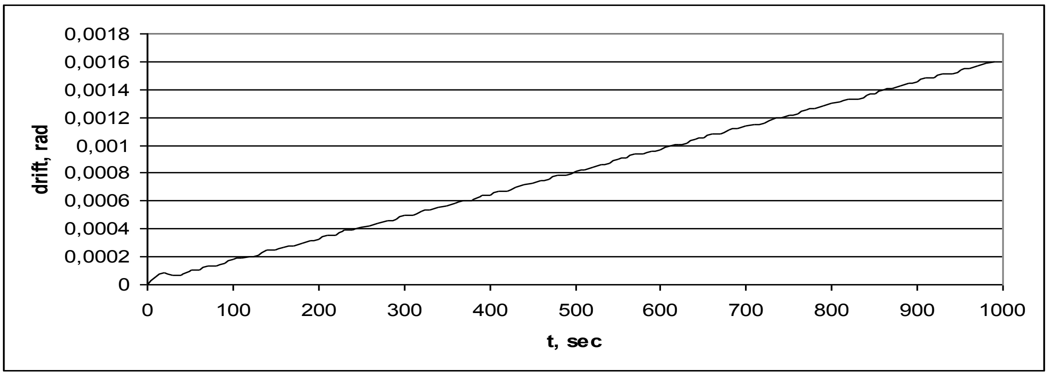
Let us consider another set of the values of α and β in the Miller formula (2). Let us build the new algorithm based on the fact that $\alpha + \beta = \frac{90}{80}$. To this end we consider $\alpha = 0$, $\beta = \frac{90}{80}$, i.e. the equation (2) is rewritten in the following form:

$$\vec{\theta}_n = \vec{\theta}_n^* + \left(\frac{9}{8}\right) \cdot \vec{\theta}_n^2 \times (\vec{\theta}_n^3 - \vec{\theta}_n^1). \quad (8)$$

Let us estimate the cumulative drift error for the quaternion finding algorithm, if the orientation vector increment on a time of computation is obtained from (8). The dependence of the drift error on time is shown in figure 3 *c*. Note that the drift error in the case $\alpha = 0$, $\beta = \frac{90}{80}$ in equation (2) is clearly smaller than the drift error in both the standard Miller algorithm and the optimized Miller algorithm.



a



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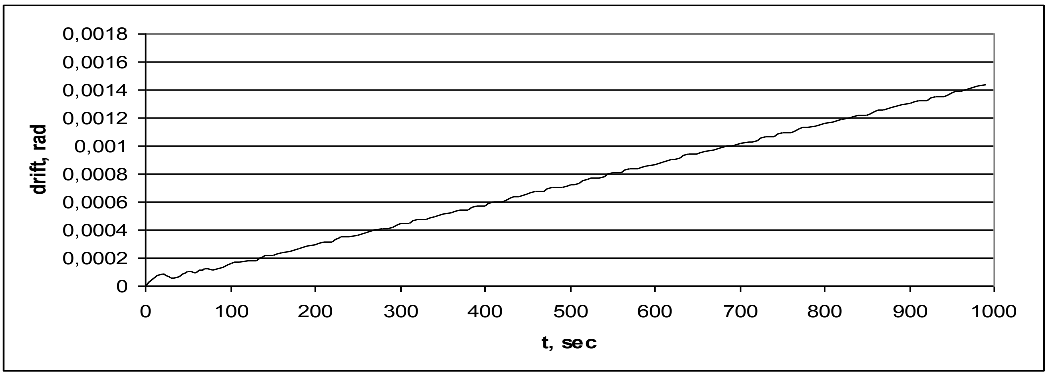

c

Figure 3. The dependence of the drift error on time for the algorithm of the fourth order in the cases: a - $\alpha = \frac{33}{80},\ \beta = \frac{57}{80}$; b - $\alpha = \frac{36}{80},\ \beta = \frac{54}{80}$; c – $\alpha = 0,\ \beta = \frac{90}{80}$.

Table 1 contains the final values of the cumulative drift errors obtained for the Miller algorithm with $\alpha = \frac{33}{80},\ \beta = \frac{57}{80}$, for the Miller algorithm optimized for the conic motion ($\alpha = \frac{36}{80},\ \beta = \frac{54}{80}$), and for the algorithm (8).

Table 1.The final values of the cumulative drift error

The Miller algorithm	Drift, rad
$\alpha = \frac{33}{80},\ \beta = \frac{57}{80}$	$1.616 \cdot 10^{-3}$
$\alpha = \frac{36}{80},\ \beta = \frac{54}{80}$	$1.601 \cdot 10^{-3}$
$\alpha = 0,\ \beta = \frac{90}{80}$	$1.433 \cdot 10^{-3}$

Conclusions

The new analytical model of a rigid body rotation on the base on four sequential turns by linear angles is proposed. In contrast to the known cases of analytical integrated systems of equations of a rigid body rotation, this model can be considered as a more generalized case of a rotation motion. The proposed model is used as the base for the reference model. For this purpose, it has been supplemented with the analytical expressions for the quasi-coordinates. The simulation of the four-frequency is used as the test motion for the estimation of the drift error of the fourth order algorithm of orientation finding with the use of the orientation vector, which increment is obtained on a time of computation using the Miller algorithm, as the intermediate parameters. It is shown that for the Miller algorithm with the coefficient values $\alpha = 0$, $\beta = \frac{90}{80}$ the cumulative drift error is less than for both the traditional algorithm and the optimized for the conic motion algorithm.

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