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Comparative analysis of polynomial and trigonometric splines

We considered classes of simple polynomial and simple trigonometric splines represented by the Fourier series. It is shown that the class of simple trigonometric splines includes the class of simple polynomial splines. For some parameter values, polynomial splines coincide with trigonometric splines; it allows transferring all the results obtained for polynomial splines to such trigonometric splines.

Introduction.

Approximation, respectively, the representation of a known or unknown function through a set of special functions can be considered a central topic of analysis; such special functions are well defined, easily calculated, and have certain analytical properties [1]. Algebraic and trigonometric polynomials, exponential functions, polynomial [2] and trigonometric [3] splines, are often used as special functions.

Let's define as C_p^r the set of 2π - periodic, r times continuously differentiable on $[0,2\pi)$ functions, and as C_p^{-1} – the set of 2π - periodic, piecewise-continuous functions with discontinuities of the first kind. Note that another notation system is also used.

It is known [4,5] that the best approach for approximating C_p^r class functions is using simple polynomial splines. The theory of such splines is well developed (see [3], [5]). The main disadvantage of polynomial splines, in our opinion, is that they have a piecewise structure; this leads to the fact that in practice it is needed to mainly use the splines of the third degree, which are stitched together from pieces of algebraic polynomials of the third degree. However, this spline structure significantly limits their use in many problems of computational mathematics.

Another principle of constructing splines was proposed in [3], [7], [8], namely, their presentation by uniformly convergent trigonometric series (Fourier series), the coefficients of which have a certain descending order. The indisputable advantage of such splines is that they are represented by a single expression over the entire function specification interval.

One of the properties of such trigonometric splines is that at certain parameter values they coincide with simple polynomial splines [8]; therefore, in this case, all the results regarding approximation estimates obtained for polynomial splines can be transferred to trigonometric splines.

Purpose of work.

Comparative analysis of interpolating polynomial and trigonometric simple splines classes.

Main part.

Let’s set a segment $[a,b]$ and a partition $\Delta_N = \{x_i\}_{i=1}^{N}$, $a = x_1 < x_1 < ... < x_N = b$ (N – natural) is set on this segment. Let’s also set a sequence of values $f_i = f(x_i)$ of a function $f(x) \in C_{[a,b]}^{r}$.

Let’s consider the problem of interpolation in such a setting. The task is to find a function $S(x)$, $S(x) \in C_{[a,b]}^{r}$, that satisfies the conditions $S(x_i) = f(x_i)$, $i = 1, 2, ..., N$.

The main difficulty lies in constructing the function $S(x)$ being r times continuously differentiable on $[a,b]$, which depends on N parameters. Currently, such construction can be done in two approaches.

The first approach is to construct the function $S(x) \in C_{[a,b]}^{r}$ by the method of piecewise polynomial or piecewise trigonometric interpolation. In the case of piecewise polynomial interpolation, we come to a well-developed theory of polynomial splines (we will only mention the classical works of Ahlberg, Stechkin, Zavyalov, Tikhomirov, Korneychuk, and many others). In the case of piecewise trigonometric interpolation, we obtain Schoenberg splines [6], the theory of which is underdeveloped. It should be noted that the authors are not aware of attempts to stitch together generalized polynomials composed of other classes’ functions.

With another approach, the function is constructed as a sum of infinite, uniformly converging trigonometric Fourier series. This method is based on the theory of generalized trigonometric functions [8], which are a generalization of L.Schwartz's periodic functions (distributions) [9]. For some values of the parameters defining these trigonometric series, both approaches lead to the same results; consequently, the functions obtained this way can be called trigonometric splines. In addition, this fact allows us to transfer to such trigonometric splines all the results of the approximation theory obtained for polynomial splines.

As a result, it is possible to combine two powerful theories – the theory of polynomial splines and the theory of trigonometric Fourier series; this gives reason to expect the emergence of new results of both theoretical and practical nature.

Both methods of constructing polynomial and trigonometric splines have their advantages and disadvantages.

Hereinafter we will consider only simple polynomial splines, that is, splines of defect 1 [2]. A simple spline of the r-th power belongs to $C_{[a,b]}^{r-1}$.

Let's consider in more detail the main properties of polynomial and trigonometric splines, which are related to the methods of their construction; note that the order of these properties was randomly chosen by the authors.

Periodicity.

When constructing polynomial splines, there are periodic and non-periodic splines; this difference is considered by setting certain boundary conditions. However, using the method of phantom nodes of periodic continuation of functions

$f(x) \in C^k_{[a,b]}$ proposed by the authors, non-periodic splines can easily be reduced to periodic ones [11]. The advantage of such a reduction is explained by the fact that the fundamental results of the theory of approximations regarding polynomial splines were obtained for the periodic case exactly [4].

Trigonometric splines are periodic by construction and belong to the set $C^k_{[a,b]}$; however, they can also be used to interpolate non-periodic functions [11].

Convergence multipliers.

When constructing trigonometric splines, it is necessary to specify the type of convergence multipliers, which have $O(n^{-(1+r)})$ order of decreasing, in which the sum of the trigonometric series belongs to the set $C^{r-1}_{[0,2\pi]}$. Polynomial analogs of trigonometric splines exist in many cases, for example, in the case when the convergence multipliers have the form $k^{-(1+r)}$. Hereinafter we will consider only similar cases.

Degrees of splines.

When constructing simple polynomial splines, it is necessary to solve systems of linear equations with strip matrices, the complexity of which increases sharply with increasing degrees of splines; this leads to the fact that splines of the third degree are most often used in practice.

When constructing trigonometric splines, there are no restrictions on their degree.

Parity of degrees.

When constructing polynomial splines, it is necessary to distinguish between even and odd values of the spline degrees, since the algorithms for constructing such splines differ.

When constructing trigonometric splines, there is no need to distinguish spline degrees, since the algorithms for constructing such splines are similar.

Uniform stitching grids and interpolation grids.

When constructing polynomial splines of odd degrees, the stitching grid is simultaneously the interpolation grid; when constructing polynomial splines of even degrees, the stitching grid differs from the interpolation grid.

When constructing trigonometric splines, the stitching grid and the interpolation grid can either match or differ; this is set by the parameters of the trigonometric spline. As a result, not all trigonometric splines have polynomial counterparts.

Non-uniform stitching grids and interpolation grids.

Constructing polynomial splines on non-uniform grids does not cause any particular difficulties.

The construction of trigonometric splines on non-uniform grids was not considered by the authors.

Construction algorithms.

As we have already said, when constructing polynomial splines, it is necessary to solve systems of linear equations with strip matrices, the complexity of which increases sharply as the degree of splines increases.

When constructing trigonometric splines, it is necessary to calculate discrete Fourier coefficients, and the well-known Fast Fourier Transform (FFT) algorithms can be used. When constructing trigonometric splines, there are no restrictions on their degree.

Fundamental interpolation splines.

The construction of fundamental interpolation polynomial splines is rather complicated; such splines have mainly theoretical applications.

The construction of fundamental interpolation trigonometric splines does not require extra difficulties; such splines can be recommended for wide application in practice.

Fundamental approximation splines.

The authors are not familiar with fundamental approximating polynomial splines.

Fundamental approximation trigonometric splines are constructed naturally, based on the approximation of fundamental trigonometric splines.

B-splines.

Polynomial B-splines are constructed sequentially with increasing degrees either by convolution or based on recurrence relations.

Trigonometric B-splines are constructed as sums of rapidly converging trigonometric series; at the same time, there is no need for their sequential construction with increasing degrees.

Differentiation and integration.

As we have already said, the main disadvantage of polynomial splines is their piecewise polynomial structure; accordingly, operations of differentiation and integration of such splines lead to piecewise polynomial functions.

The operations of differentiating and integrating trigonometric splines are performed much simpler, namely by term-wise differentiation or integration of uniformly convergent trigonometric series.

Conclusions.

1. It is shown that the class of trigonometric splines is presented by the Fourier series.
2. It is possible to combine two powerful theories – the theory of polynomial splines and the theory of trigonometric Fourier series; this gives reason to expect new results of both theoretical and practical nature.
3. The set of simple polynomial splines is a subset of the set of trigonometric splines.
4. For some parameter values, trigonometric splines coincide with simple polynomial splines; all the results of the approximation theory obtained for simple polynomial splines can be transferred to such trigonometric splines.
5. When constructing trigonometric splines, FFT algorithms are widely used.

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